ARPREC: An Arbitrary Precision Computation Package

David H. Bailey
Lawrence Berkeley National Laboratory
Berkeley, CA, USA 94720
dhbailey@lbl.gov

Collaborators:
David J. Broadhurst, Open University, UK
Yozo Hida, University of California, Berkeley
Xiaoye S. Li, Lawrence Berkeley National Laboratory, Berkeley
Brandon Thompson, University of California, Berkeley

Types of High Precision Arithmetic

- Double precision (16 digits): Conventional 64-bit IEEE arithmetic.
- Double-Double (32 digits): Can be done by using IEEE arithmetic techniques. Approx. 5 times as expensive as DP.
- Quad-Double (64 digits): Can be done using IEEE arithmetic techniques. Approx. 5 times as expensive as DD.
- Arbitrary precision (100 to millions of digits): Requires arbitrary precision arithmetic software.

Integer Relation Detection

Given a real or complex vector $x = (x_1, x_2, \cdots, x_n)$ an *integer relation* (IR) algorithm seeks integers a_i, not all zero, such that

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

to within the available numerical accuracy.

- Original IR algorithm found in 1977 by Helaman Ferguson and Rodney Forcade.
- Current state of art: Ferguson's "PSLQ" algorithm — recently named one of ten "algorithms of the century" by *Computing in Science and Engineering*.
- Very high numeric precision (hundreds or thousands of digits) must be employed in integer relation calculations.

Applications of PSLQ: Recognizing Numeric Constants

If α is algebraic of degree n, the polynomial satisfied by α can be found by computing the vector $(1, \alpha, \alpha^2, \cdots, \alpha^n)$ to high precision, and then applying PSLQ.

Chaos theory example:

Let $B_3 = 3.54409035955\cdots$ be the third bifurcation point of the logistic map $x_{k+1} = r x_k (1 - x_k)$. In other words, B_3 is the smallest r such that successive iterates x_k exhibit eight-way periodicity instead of four-way periodicity.

Computations using a predecessor algorithm to PSLQ found that B_3 is a root the polynomial

$$0 = 4913 + 2108t^2 - 604t^3 - 977t^4 + 8t^5 + 44t^6 + 392t^7 - 193t^8 - 40t^9 + 48t^{10} - 12t^{11} + t^{12}$$

Recently a PSLQ program found that $\alpha = -B_4(B_4 - 2)$ satisfies a 120-degree polynomial, so that B_4 satisfies a 240-degree polynomial.

Applications of PSLQ: Euler Sums

Let $\zeta(t) = \sum_{j=1}^{\infty} j^{-t}$ be the Riemann zeta function, and $\mathrm{Li}_n(x) = \sum_{j=1}^{\infty} x^j j^{-n}$ the polylogarithm function. The following were found using PSLQ computations:

$$\begin{aligned}
\sum_{k=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)^2 (k+1)^{-4} &= \frac{37}{22680}\pi^6 - \zeta^2(3) \\
\sum_{k=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)^3 (k+1)^{-6} &= \zeta^3(3) + \frac{197}{24}\zeta(9) + \frac{1}{2}\pi^2\zeta(7) \\
&\quad - \frac{11}{120}\pi^4\zeta(5) - \frac{37}{7560}\pi^6\zeta(3) \\
\sum_{k=1}^{\infty} \left(1 - \frac{1}{2} + \cdots + (-1)^{k+1}\frac{1}{k}\right)^2 (k+1)^{-3} &= 4\,\mathrm{Li}_5\left(\frac{1}{2}\right) - \frac{1}{30}\ln^5(2) - \frac{17}{32}\zeta(5) \\
&\quad - \frac{11}{720}\pi^4\ln(2) + \frac{7}{4}\zeta(3)\ln^2(2) \\
&\quad + \frac{1}{18}\pi^2\ln^3(2) - \frac{1}{8}\pi^2\zeta(3)
\end{aligned}$$

Applications of PSLQ: Apery Sums

It has been known for some time, through the research of Apery, that

$$\begin{aligned}\zeta(2) &= 3\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}\\ \zeta(3) &= \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}\\ \zeta(4) &= \frac{36}{17}\sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}}\end{aligned}$$

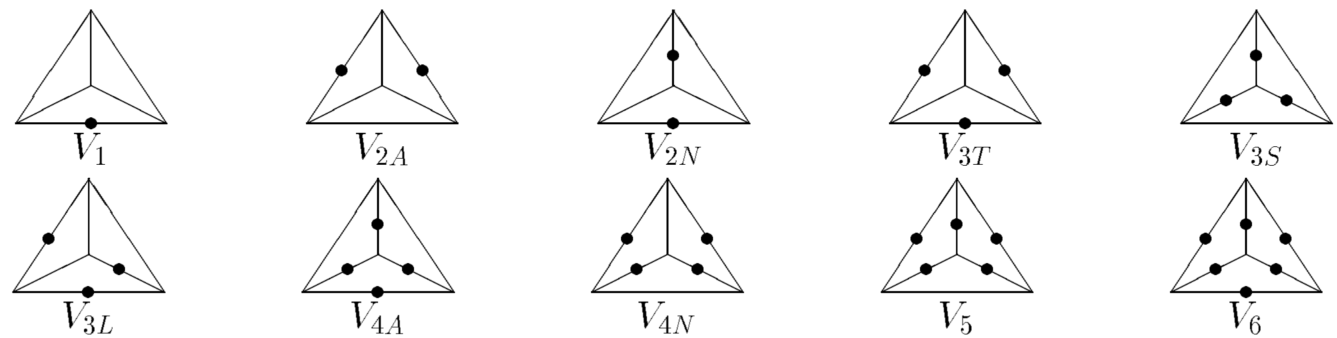
These results have led many to suggest that

$$S(n) = \sum_{k>0}\frac{1}{k^n\binom{2k}{k}},$$

for $n > 4$, might be a simple constant. It has now been shown that $S(n)$ can be expressed in terms of the Riemann zeta function $\zeta(n)$ and Clausen's function $M(a,b)$. A sample evaluation is

$$\begin{aligned}S(9) &= \pi\left[2M(7,1)+\frac{8}{3}M(5,3)+\frac{8}{9}\zeta(2)M(5,1)\right]-\frac{13921}{216}\zeta(9)\\ &\quad+\frac{6211}{486}\zeta(7)\zeta(2)+\frac{8101}{648}\zeta(6)\zeta(3)+\frac{331}{18}\zeta(5)\zeta(4)-\frac{8}{9}\zeta^3(3)\end{aligned}$$

Ten Tetrahedral Cases from Quantum Field Theory



Evaluations of constants associated with the ten cases:

$$
\begin{aligned}
V_1 &= 6\zeta(3) + 3\zeta(4) \\
V_{2A} &= 6\zeta(3) - 5\zeta(4) \\
V_{2N} &= 6\zeta(3) - \frac{13}{2}\zeta(4) - 8U \\
V_{3T} &= 6\zeta(3) - 9\zeta(4) \\
V_{3S} &= 6\zeta(3) - \frac{11}{2}\zeta(4) - 4C^2 \\
V_{3L} &= 6\zeta(3) - \frac{15}{4}\zeta(4) - 6C^2 \\
V_{4A} &= 6\zeta(3) - \frac{77}{12}\zeta(4) - 6C^2 \\
V_{4N} &= 6\zeta(3) - 14\zeta(4) - 16U \\
V_5 &= 6\zeta(3) - \frac{469}{27}\zeta(4) + \frac{8}{3}C^2 - 16V \\
V_6 &= 6\zeta(3) - 13\zeta(4) - 8U - 4C^2
\end{aligned}
$$

$$
\begin{aligned}
U &= \sum_{j>k>0} \frac{(-1)^{j+k}}{j^3 k} \\
C &= \sum_{k>0} \sin(\pi k/3)/k^2 \\
V &= \sum_{j>k>0} (-1)^j \cos(2\pi k/3)/(j^3 k)
\end{aligned}
$$

A Quadrature Example

Using a high-precision numerical quadrature program, Jon Borwein, Greg Fee and DHB observed that if

$$C(a) \;=\; \int_0^1 \frac{\arctan(\sqrt{x^2+a^2})\,dx}{\sqrt{x^2+a^2}(x^2+1)}$$

then

$$\begin{aligned} C(0) &= \pi \log 2/8 + G/2 \\ C(1) &= \pi/4 - \pi\sqrt{2}/2 + 3\sqrt{2}\arctan(\sqrt{2})/2 \\ C(\sqrt{2}) &= 5\pi^2/96 \end{aligned}$$

where G is Catalan's constant (the third result appeared in the *MAA Monthly*, Aug/Sept 2002). These particular results have now led to several general results, including:

$$\int_0^\infty \frac{\arctan(\sqrt{x^2+a^2})\,dx}{\sqrt{x^2+a^2}(x^2+1)} \;=\; \frac{\pi}{2\sqrt{a^2-1}}\left[2\arctan(\sqrt{a^2-1}) - \arctan(\sqrt{a^4-1})\right]$$

Peter Borwein's Observation on the Binary Digits of log 2

In 1995, Peter Borwein observed that an arbitrary binary digit of $\log 2$ can be calculated by using a very simple algorithm:

Let $\{\cdot\}$ denote the fractional part. Then we can write

$$\begin{aligned}
\{2^d \log 2\} &= \left\{2^d \sum_{k=1}^{\infty} \frac{1}{k 2^k}\right\} = \left\{\sum_{k=1}^{\infty} \frac{2^{d-k}}{k}\right\} \\
&= \left\{\sum_{k=1}^{d} \frac{2^{d-k}}{k}\right\} + \left\{\sum_{k=d+1}^{\infty} \frac{2^{d-k}}{k}\right\} \\
&= \left\{\sum_{k=1}^{d} \frac{2^{d-k} \bmod k}{k}\right\} + \left\{\sum_{k=d+1}^{\infty} \frac{2^{d-k}}{k}\right\}
\end{aligned}$$

- The numerators $2^{d-k} \bmod k$ can be very rapidly evaluated using the binary algorithm for exponentiation performed modulo k.
- Only a few terms of the second summation need be evaluated.
- All computations can be done with ordinary 64-bit floating-point arithmetic.

A More General Result

Any constant α given by a formula of the type

$$\alpha \;=\; \sum_{k=0}^{\infty} \frac{p(k)}{b^k q(k)}$$

(where $p(k)$ and $q(k)$ are integer polynomials, $\deg p < \deg q$ and q has no zeroes for positive k) has the rapid individual digit computation property.

Is there a formula of this type for π? None was known in 1995.

The BBP Formula for π

By applying DHB's PSLQ computer program to set of computed constants for which formulas of this type were known, with the numerical value of π appended, this formula was found for π:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right)$$

Proof: An exercise in calculus.

Question: Why wasn't this formula discovered 250 years ago?

Some Other Constants with Base 2 BBP-Type Formulas

$$\begin{aligned}
\log 3 &= \sum_{k=0}^{\infty} \frac{1}{4^k(2k+1)} \\
\log 7 &= \frac{3}{4} \sum_{k=0}^{\infty} \frac{1}{8^k} \left(\frac{2}{8k+1} + \frac{1}{8k+2} \right) \\
\pi^2 &= \frac{1}{8} \sum_{k=0}^{\infty} \frac{1}{64^k} \left(\frac{144}{(6k+1)^2} - \frac{216}{(6k+2)^2} - \frac{72}{(6k+3)^2} - \frac{54}{(6k+4)^2} + \frac{9}{(6k+5)^2} \right) \\
\log^2 2 &= \frac{1}{6} \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{16}{(8k+1)^2} - \frac{40}{(8k+2)^2} - \frac{8}{(8k+3)^2} - \frac{28}{(8k+4)^2} \right. \\
&\quad \left. - \frac{4}{(8k+5)^2} - \frac{10}{(8k+6)^2} + \frac{2}{(8k+7)^2} - \frac{3}{(8k+8)^2} \right) \\
\pi^2 - 6 \log^2 2 &= 12 \sum_{k=1}^{\infty} \frac{1}{k^2 2^k} \\
\pi \sqrt{3} &= \frac{9}{32} \sum_{k=0}^{\infty} \frac{1}{64^k} \left(\frac{16}{6k+1} - \frac{8}{6k+2} - \frac{2}{6k+4} - \frac{1}{6k+5} \right)
\end{aligned}$$

An Arctan Formula

$$
\begin{aligned}
\tan^{-1}\left(\frac{4}{5}\right) &= \frac{1}{2^{17}}\sum_{k=0}^{\infty}\frac{1}{2^{20k}}\left(\frac{524288}{40k+2}-\frac{393216}{40k+4}-\frac{491520}{40k+5}+\frac{163840}{40k+8}\right.\\
&\quad+\frac{32768}{40k+10}-\frac{24576}{40k+12}+\frac{5120}{40k+15}+\frac{10240}{40k+16}+\frac{2048}{40k+18}\\
&\quad+\frac{1024}{40k+20}+\frac{640}{40k+24}+\frac{480}{40k+25}+\frac{128}{40k+26}-\frac{96}{40k+28}\\
&\quad\left.+\frac{40}{40k+32}+\frac{8}{40k+34}-\frac{5}{40k+35}-\frac{6}{40k+36}\right)
\end{aligned}
$$

Similar formulas have been found for arctans of numerous other rational arguments.

Some Base 3 BBP-Type Formulas

$$\begin{aligned}
\log 2 &= \frac{2}{27}\sum_{k=0}^{\infty}\frac{1}{81^k}\left(\frac{9}{4k+1}+\frac{1}{4k+3}\right)\\
&= \sum_{n=0}^{\infty}\frac{1}{9^n(2n-1)}\\
\pi^2 &= \frac{2}{27}\sum_{k=0}^{\infty}\frac{1}{729^k}\left(\frac{243}{(12k+1)^2}-\frac{405}{(12k+2)^2}-\frac{81}{(12k+4)^2}-\frac{27}{(12k+5)^2}\right.\\
&\quad\left.-\frac{72}{(12k+6)^2}-\frac{9}{(12k+7)^2}-\frac{9}{(12k+8)^2}-\frac{5}{(12k+10)^2}+\frac{1}{(12k+11)^2}\right)\\
6\sqrt{3}\tan^{-1}\left(\frac{\sqrt{3}}{7}\right) &= \sum_{k=0}^{\infty}\frac{1}{27^k}\left(\frac{3}{3k+1}+\frac{1}{3k+2}\right)
\end{aligned}$$

Normality

The real number α is *normal* to base b if every sequence of m digits in the base-b expansion of α appears with limiting frequency b^{-m}.

Widely believed to be normal base b for all bases b:

- π and e.
- $\log 2$ and $\sqrt{2}$.
- The golden mean $\tau = (1 + \sqrt{5})/2$.
- *Every* irrational algebraic number.
- Many other "natural" irrational constants.

But there are *no* proofs for any of these constants, for any base.

Normality proofs exist only for handful of artifically constructed constants, such as Champernowne's number: 0.1234567891011121314...

A Connection Between BBP-Type Formulas and Normality

Theorem: The BBP-type constant

$$\alpha = \sum_{k=0}^{\infty} \frac{p(k)}{b^k q(k)}$$

(where $p(k)$ and $q(k)$ are integer polynomials, $\deg p < \deg q$ and q has no zeroes for positive k) is normal base b if and only if the sequence $x_0 = 0$, and

$$x_n = \left(b x_{n-1} + \frac{p(n)}{q(n)}\right) \bmod 1$$

is equidistributed in the unit interval.

Proof Sketch: Let α_n be the base-b expansion of α after the n-th digit. Following the BBP approach, we can write

$$\begin{aligned} \alpha_n &= \left\{\sum_{k=0}^{n} \frac{b^{n-k} p(k)}{q(k)}\right\} + \left\{\sum_{k=n+1}^{\infty} \frac{b^{n-k} p(k)}{q(k)}\right\} \\ &= \left(b\alpha_{n-1} + \frac{p(n)}{q(n)}\right) \bmod 1 \quad + E_n \end{aligned}$$

where E_n goes to zero.

Two Examples

1. Let $x_0 = 0$, and

$$x_n = \left(2x_{n-1} + \frac{1}{n}\right) \bmod 1$$

   Is (x_n) equidistributed in $[0, 1)$?

2. Let $x_0 = 0$ and

$$x_n = \left(16x_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21}\right) \bmod 1$$

   Is (x_n) equidistributed in $[0, 1)$?

If answer to Question 1 is "yes", then $\log 2$ is normal to base 2.

If answer to Question 2 is "yes", then π is normal to base 16 (and hence to base 2 also).

A Class of Provably Normal Constants

Using the BBP approach, Richard Crandall and DHB have now proven normality for a class of constants, the simplest instance of which is

$$\begin{aligned}\alpha_{2,3} &= \sum_{k=1}^{\infty} \frac{1}{3^k 2^{3^k}} \\ &= 0.041883680831502985071252898624571682426096\ldots_{10} \\ &= \mathtt{0.0AB8E38F684BDA12F684BF35BA781948B0FCD6E9E0}\ldots_{16}.\end{aligned}$$

$\alpha_{2,3}$ was actually proven normal base 2 in a little-known paper by Stoneham in 1977. Crandall and DHB proved normality and transcendence for an uncountably infinite class that includes $\alpha_{2,3}$.

These constants also possess the rapid individual digit computation property. The googol-th binary digit of $\alpha_{2,3}$ is zero.

Overview of the ARPREC Package

- Based on earlier MPFUN-77 and MPFUN-90 Fortran packages.
- Code written in C++ for high performance and broad portability.
- C++ and Fortran-90 translation modules that permit conventional C++ and Fortran-90 programs to utilize the package with only very minor changes to source code.
- Arbitrary precision integer, floating and complex datatypes.
- Support for datatypes with differing precision levels.
- Inter-operability with conventional integer and floating-point datatypes.
- Common transcendental functions (sqrt, exp, sin, etc).
- Quadrature routines (for numerical integration).
- PSLQ routines (for integer relation detection).
- Special routines for extra-high precision ($>$ 1000 digits) computation.

Structure of ARPREC Multiprecision Data

- An array of 64-bit IEEE floats.
- Word 1: Number of words allocated for array.
- Word 2: Number of mantissa words used; sign is sign of number.
- Word 3: Exponent (powers of 2^{48}).
- Word 4 through $n+3$: Mantissa words, each in the range $[0, 2^{48})$.
- Word $n+4$ and $n+5$: For convenience in arithmetic routines.

Exact Arithmetic on 64-Bit IEEE Data

Double + double.

1. $s \leftarrow a \oplus b$
2. $v \leftarrow s \ominus a$
3. $e \leftarrow (a \ominus (s \ominus v)) \oplus (b \ominus v)$

Split.

1. $t \leftarrow (2^{27} + 1) \otimes a$
2. $a_{\mathrm{hi}} \leftarrow t \ominus (t \ominus a)$
3. $a_{\mathrm{lo}} \leftarrow a \ominus a_{\mathrm{hi}}$

Double × double (not needed if system has fused multiply-add).

1. $p \leftarrow a \otimes b$
2. $(a_{\mathrm{hi}}, a_{\mathrm{lo}}) \leftarrow \textsc{Split}(a)$
3. $(b_{\mathrm{hi}}, b_{\mathrm{lo}}) \leftarrow \textsc{Split}(b)$
4. $e \leftarrow ((a_{\mathrm{hi}} \otimes b_{\mathrm{hi}} \ominus p) \oplus a_{\mathrm{hi}} \otimes b_{\mathrm{lo}} \oplus a_{\mathrm{lo}} \otimes b_{\mathrm{hi}}) \oplus a_{\mathrm{lo}} \otimes b_{\mathrm{lo}}$

Normalize result (two words with 48 bits each).

1. $p' \leftarrow (p/2^{48} \oplus 2^{52}) \ominus 2^{52}$
2. $e' \leftarrow (p - p') \oplus e$

Performing High-Precision Multiplications Using FFTs

If $a = (a_j,\ j = 0, 1, \cdots, n-1)$ and $b = (b_j,\ j = 0, 1, \cdots, n-1)$ are two high-precision numbers, then their $2n$-long product (except for release of carries) is merely the acyclic convolution of a and b:

Extend a and b by n zeroes, then compute:

$$c_k = \sum_{j=0}^{2n-1} a_j b_{k-j}$$

where the subscript $k-j$, if negative, is taken to be $k-j+2n$. These results can thus be rapidly computed using an FFT:

$$c_k = F_k^{-1}[F_k(a) \cdot F_k(b)]$$

where for example

$$F_k(a) \; = \; \sum_{j=0}^{2n-1} a_j e^{-2\pi i jk/(2n)}$$

Additional time can be saved by using real-to-complex and complex-to-real FFTs.

Improvements from MPFUN to ARPREC

- Improved arithmetic performance: Schemes on previous viewgraph can be performed as register operations, which are very fast on modern RISC systems.
- Taylor's series routines: Routines for sine, cosine and exponential reduce precision as the size of terms decreases.
- Sine/Cosine: Argument is reduced to the nearest multiple of $\pi/256$, instead of $\pi/16$.
- FFT-based multiplication: A radix-four FFT algorithm is used, instead of a radix-two FFT.

Arithmetic Performance: ARPREC vs MPFUN

Arithmetic test loop (400 decimal digit precision):

- MPFUN = 14.78 seconds.
- ARPREC = 10.80 seconds.

Polylogarithmic ladder calculation (50,000 digit precision):

- MPFUN = 1408 CPU-hours on Cray T3E parallel system (32 CPUs).
- ARPREC = 1062 CPU-hours on IBM SP parallel system (64 CPUs).

Three State-of-the-Art Quadrature Routines

- Gaussian quadrature.
- Error function quadrature.
- Tanh-sinh quadrature.

High-Precision Gaussian Quadrature

An integral on $[-1, 1]$ is approximated as the sum

$$\int_{-1}^{1} f(x)\, dx \approx \sum_{j=0}^{n} w_j f(x_j),$$

where the abscissas x_j are the roots of the n-th degree Legendre polynomial $P_n(x)$ on $[-1, 1]$, and the weights x_j are

$$w_j = \frac{-2}{(n+1)P'_n(x_j)P_{n+1}(x_j)}$$

The abscissas x_j are computed using a Newton iteration scheme, with starting value $\cos[\pi(j - 1/4)/(n + 1/2)]$. The function $P_n(x)$ is computed using an n-long iteration of the recurrence $P_0(x) = 0$, $P_1(x) = 1$ and

$$(k+1)P_{k+1}(x) = (2k+1)xP_k(x) - kP_{k-1}(x)$$

for $k \geq 2$. The derivative $P'_n(x)$ is computed as

$$P'_n(x) = n(xP_n(x) - P_{n-1}(x))/(x^2 - 1)$$

The Euler-Maclaurin Formula and High-Precision Quadrature

Let $h = (b-a)/n$ and $x_j = a + jh$. Then

$$\begin{aligned}\int_a^b f(x)\,dx \;&=\; \frac{h}{2}\left(f(a)+f(b)\right) + h\sum_{j=1}^{n-1} f(x_j) \\ &+ \sum_{i=1}^{m} \frac{h^{2i}B_{2i}}{(2i)!}\left(f^{(2i-1)}(b) - f^{(2i-1)}(a)\right) + E\end{aligned}$$

where the error term E is smaller than $h^{2m+1}/(2m+2)!$ times a certain definite integral that does not depend on n or h.

The E-M formula also applies for functions defined on an infinite interval, where $f(x)$ and all its derivatives tend rapidly to zero for large x. In this case, we have

$$\int_{-\infty}^{\infty} f(x)\,dx \;=\; h\sum_{j=-\infty}^{\infty} f(x_j) + E$$

where the error term E tends to zero more rapidly than any power of h, as h is decreased.

The Error Function Quadrature Scheme

Let $g(x) = \mathrm{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-t^2}\, dt$. Note that $\mathrm{erf}(x)$ ranges monotonically from -1 to 1. Thus we can write

$$\int_{-1}^{1} f(x)\, dx \;=\; \int_{-\infty}^{\infty} f(g(t)) g'(t)\, dt$$

Since $g'(t) = 2/\sqrt{\pi} \cdot e^{-t^2}$ goes to zero rapidly for large t, the integrand on the RHS is, for many $f(x) \in C^\infty(-1,1)$, a nice bell-shaped curve for which the E-M formula applies. Thus we can write

$$\int_{-1}^{1} f(x)\, dx \;\approx\; h \sum_{k=-\infty}^{\infty} f(g(kh)) g'(kh) \;\approx\; h \sum_{k=-N}^{N} f(x_k) w_k$$

where $x_k = \mathrm{erf}(kh)$ and $w_k = 2/\sqrt{\pi} \cdot e^{-(kh)^2}$. The x_k and w_k values can be pre-computed.

The tanh-sinh quadrature scheme uses the function $g(t) = \tanh(\pi/2 \cdot \sinh t)$, whose derivatives tend to zero even faster.

Test Problems for Quadrature Routines

Well-behaved continuous functions on finite itervals:

$$1: \int_0^1 t\log(1+t)\,dt = 1/4 \qquad 2: \int_0^1 t^2 \arctan t\,dt = (\pi - 2 + 2\log 2)/12$$

$$3: \int_0^{\pi/2} e^t \cos t\,dt = (e^{\pi/2}-1)/2 \qquad 4: \int_0^1 \frac{\arctan(\sqrt{2+t^2})}{(1+t^2)\sqrt{2+t^2}}\,dt = 5\pi^2/96$$

Continuous functions on finite itervals, but with a vertical derivative at an endpoint:

$$5: \int_0^1 \sqrt{t}\log t\,dt = -4/9 \qquad 6: \int_0^1 \sqrt{1-t^2}\,dt = \pi/4$$

Functions on finite intervals with an integrable singularity at an endpoint:

$$7: \int_0^1 \frac{t}{\sqrt{1-t^2}}dt = 1 \qquad 8: \int_0^1 \log t^2\,dt = 2$$

$$9: \int_0^{\pi/2} \log(\cos t)\,dt = -\pi\log(2)/2 \qquad 10: \int_0^{\pi/2} \sqrt{\tan t}\,dt = \pi\sqrt{2}/2$$

Functions on an infinite interval:

$$11: \int_0^\infty \frac{1}{1+t^2}\,dt = \pi/2 \qquad 12: \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,dt = \sqrt{\pi}$$

$$13: \int_0^\infty e^{-t^2/2}\,dt = \sqrt{\pi/2}$$

Oscillatory functions on an infinite interval:

$$14: \int_0^\infty e^{-t}\cos t\,dt = 1/2 \qquad 15: \int_0^\infty \frac{\sin t}{t}\,dt = \pi/2$$

Performance of Quadrature Routines on Test Problems

	QUADGS			QUADERF			QUADTS		
Prob.	Level	Time	Error	Level	Time	Error	Level	Time	Error
Init	9	2755.08		9	138.81		9	48.21	
1	6	8.90	10^{-422}	9	60.43	10^{-421}	7	13.61	10^{-390}
2	6	9.35	10^{-422}	9	39.91	10^{-412}	8	23.78	10^{-421}
3	5	4.43	10^{-420}	9	48.04	10^{-419}	7	12.76	0
4	6	9.15	10^{-420}	9	100.07	10^{-409}	8	40.97	10^{-421}
5	9	79.88	10^{-11}	9	72.97	10^{-420}	7	16.70	10^{-420}
6	9	3.66	10^{-12}	9	4.00	10^{-420}	7	0.91	10^{-392}
7	9	4.45	10^{-4}	8	2.45	10^{-210}	6	0.56	10^{-196}
8	9	77.73	10^{-6}	9	69.73	10^{-415}	7	16.14	10^{-417}
9	9	103.29	10^{-7}	9	74.59	10^{-416}	7	18.94	10^{-390}
10	9	33.13	10^{-4}	8	7.96	10^{-210}	6	2.48	10^{-194}
11	6/6	0.57	0	9/9	5.38	10^{-409}	8/8	2.37	0
12	9/9	96.89	10^{-4}	8/9	63.56	10^{-133}	6/9	45.85	10^{-217}
13	5/9	48.40	10^{-364}	9/9	70.26	10^{-97}	7/9	41.89	10^{-250}
14	4/9	79.18	10^{-126}	9/9	145.89	10^{-69}	7/9	86.80	10^{-165}
15	5/9	36.22	10^{-19}	9/9	44.11	10^{-17}	7/9	33.62	10^{-19}

The Experimental Mathematician's Toolkit

- Interactive tool based on ARPREC.
- All common arithmetic expressions — use Mathematica format.
- Many common math constants — π, $e, \log 2$, Catalan's constant, Euler's gamma constant, etc.
- Many common math functions — sin, cos, exp, sqrt, erf, zeta, etc.
- Quadrature, on finite or infinite intervals (choice of 3 routines).
- Summations, with finite or infinite limits.
- PSLQ calculations (choice of 1-, 2- or 3-level multi-pair routines).

Test program now available from author.

For Full Details

- David H. Bailey, Peter B. Borwein and Simon Plouffe, "On The Rapid Computation of Various Polylogarithmic Constants," *Mathematics of Computation*, vol. 66, no. 218, 1997, pp. 903–913.
- David H. Bailey, "A Compendium of BBP-Type Formulas," 2002.
- David H. Bailey and Richard E. Crandall, "On the Random Character of Fundamental Constant Expansions," *Experimental Mathematics*, June 2001.
- David H. Bailey and Richard E. Crandall, "Random Generators and Normal Numbers," 2002.
- David H. Bailey, Yozo Hida, Xiaoye S. Li and Brandon Thompson, "ARPREC: An Arbitrary Precision Computation Package," Oct 2002.
- David H. Bailey, "A Comparison of Three High-Precision Quadrature Schemes," Oct 2002.

Papers: `http://www.nersc.gov/~dhbailey/dhbpapers`
Computer code: `http://www.nersc.gov/~dhbailey/mpdist`